Some Recent Results on Pair Correlation Functions and Susceptibilities in Exactly Solvable Models

Jacques H H Perk and Helen Au-Yang

Department of Physics, Oklahoma State University, 145 Physical Sciences, Stillwater, OK 74078-3072, USA

E-mail: `perk@okstate.edu`

Abstract.   Using detailed exact results on pair-correlation functions of Z-invariant Ising models, we can write and run algorithms of polynomial complexity to obtain wavevector-dependent susceptibilities for a variety of Ising systems. Reviewing recent work we compare various periodic and quasiperiodic models, where the couplings and/or the lattice may be aperiodic, and where the Ising couplings may be either ferromagnetic, or antiferromagnetic, or of mixed sign. We present some of our results on the square-lattice fully-frustrated Ising model. Finally, we make a few remarks on our recent works on the pentagrid Ising model and on overlapping unit cells in three dimensions and how these works can be utilized once more detailed results for pair correlations in, *e.g.*, the eight-vertex model or the chiral Potts model or even three-dimensional Yang–Baxter integrable models become available.

1. Introduction

Detailed exact results on pair-correlation functions and susceptibilities are usually hard to come by, even in models that are "exactly solvable." The best results are available for Z-invariant Ising-type models, and even though the results are often not in closed form, we may follow Tony Guttmann and claim an exact result if we can produce an algorithm of polynomial complexity to obtain detailed information on, say, wavevector-dependent susceptibilities for a variety of Ising systems.

For a long time, the paper of Wu, McCoy, Tracy and Barouch [1] has been the standard work on the pair correlations and susceptibility of the square-lattice Ising model. Only in the last few years do we know how to do much better. We now have algorithms to construct long high- and low-temperature series together with detailed expansions at criticality for the susceptibility [2, 3] using quadratic recurrence relations [4] for pair correlations. We also have much more analytic and numerical information for pair correlations and susceptibilities in more general Z-invariant inhomogeneous Ising models [5, 6, 7], together with new understanding from a field theory approach [8].

1.1. Baxter's Z-invariant inhomogeneous Ising model

We can start with an inhomogeneous Ising model on a square lattice with reduced interaction energy

$$-\beta\mathcal{H} = \sum_{m,n}(\bar{K}_{m,n}\sigma_{m,n}\sigma_{m,n+1} + K_{m,n}\sigma_{m,n}\sigma_{m+1,n})\tag{1}$$

where $\beta = 1/k_{\mathrm{B}}T$ the inverse temperature. At site (m,n) the spin takes values $\sigma_{m,n} = \pm 1$, while $K = \beta J$ and $\bar{K} = \beta\bar{J}$ are "horizontal" and "vertical" dimensionless coupling constants.

One important quantity to study is the wavevector-dependent susceptibility $\chi(\mathbf{q})$ defined by

$$k_{\mathrm{B}}T\chi(\mathbf{q}) \equiv \bar{\chi}(\mathbf{q}) = \lim_{\mathcal{L}\to\infty}\frac{1}{\mathcal{L}}\sum_{\mathbf{r}}\sum_{\mathbf{r}'}\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})}[\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle - \langle\sigma_{\mathbf{r}}\rangle\langle\sigma_{\mathbf{r}'}\rangle]\tag{2}$$

This is the Fourier transform of the connected pair correlation function. In (2), $\mathcal{L}$ is the number of lattice sites, $\mathbf{r}$ and $\mathbf{r}'$ run through all sites and $\mathbf{q} = (q_x, q_y)$ is the wavevector.

Baxter's Z-invariant Ising model [9, 10, 11] is defined in terms of oriented rapidity lines on which the rapidity variables live. (The rapidity lines can be moved at will, without changing the partition function Z, so we do not have to take a square lattice.) The faces bounded by the rapidity lines are alternatingly colored black or white and spins live somewhere on the black faces forming the Ising system $\{\sigma_{\mathbf{r}}\}$, whereas dual spins live on the white faces forming the dual Ising system $\{\sigma_{\mathbf{r}^*}^*\}$. Spins interact only with nearest neighbours from which they are separated by precisely two rapidity lines, see Fig. 1. The couplings K and $\bar{K}$ are parameterized in terms of elliptic functions of
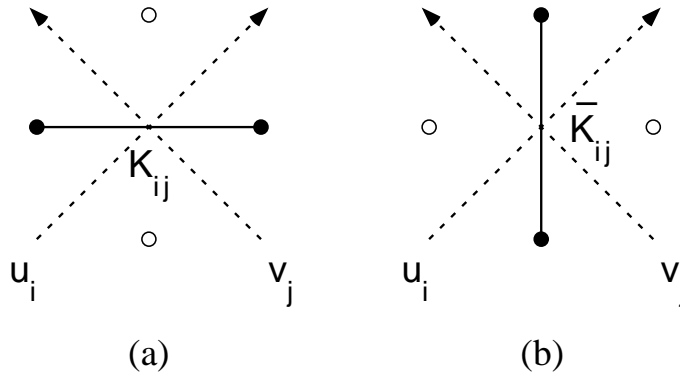

Figure 1. Two oriented rapidity lines with rapidities u_i and v_j are drawn dashed. The orientation of the coupling with respect to these rapidity lines defines whether we have (a) horizontal coupling $K_{ij} = K(u_i, v_j)$ or (b) vertical coupling $\bar{K}_{ij} = \bar{K}(u_i, v_j)$.

modulus k, *i.e.*,

$$\sinh(2K(u_1, u_2)) = k\,\mathrm{sc}(u_1 - u_2, k') = \mathrm{cs}(\mathrm{K}(k') + u_2 - u_1, k')$$

$$\sinh(2\bar{K}(u_1, u_2)) = \mathrm{cs}(u_1 - u_2, k') = k\,\mathrm{sc}(\mathrm{K}(k') + u_2 - u_1, k')\tag{3}$$

where

$$k' = \sqrt{1-k^2}, \qquad \mathrm{sc}(v,k) = \mathrm{sn}(v,k)/\mathrm{cn}(v,k) = 1/\mathrm{cs}(v,k) \qquad (4)$$

and $\mathrm{K}(k)$ is the complete elliptic integral of the first kind.

We note that K and $\bar{K}$ are interchanged if we replace u_1 by $u_2 \pm \mathrm{K}(k')$ and u_2 by u_1. In other words, flipping the orientation of a rapidity line j is equivalent to changing its rapidity variable u_j to $u_j \pm \mathrm{K}(k')$.

1.2. Two-point correlation functions

Z-invariance implies [9] that the pair correlation only depends on elliptic modulus k and the values of the $2m$ rapidity variables $u_1, \ldots, u_{2m}$ that pass between the two spins, implying the existence of an infinite set of universal functions $g_2, g_4, \ldots, g_{2m}, \ldots$ such that for any permutation P and rapidity shift v

$$\langle \sigma \sigma' \rangle = g_{2m}(k; \bar{u}_1, \ldots, \bar{u}_{2m}) = g_{2m}(k; \bar{u}_{\mathrm{P}(1)} + v, \ldots, \bar{u}_{\mathrm{P}(2m)} + v). \qquad (5)$$

Here $\bar{u}_j = u_j$ if the jth rapidity line passes between the two spins σ and σ' in a given direction and $\bar{u}_j = u_j + \mathrm{K}(k')$ if it passes in the opposite direction [11].

If two of the rapidity variables passing between the two spins differ by $\mathrm{K}(k')$, they can be viewed as belonging to a single rapidity line moving back and forth between these two spins, *i.e.* [9],

$$g_{2m+2}(k; \bar{u}_1, \ldots, \bar{u}_{2m}, \bar{u}_{2m+1}, \bar{u}_{2m+1} + \mathrm{K}(k')) = g_{2m}(k; \bar{u}_1, \ldots, \bar{u}_{2m}). \qquad (6)$$

1.3. Quadratic identity for pair correlation

For general planar Ising models we can derive a quadratic relation between pair correlation functions [4]. For the situation in Fig. 2 we have
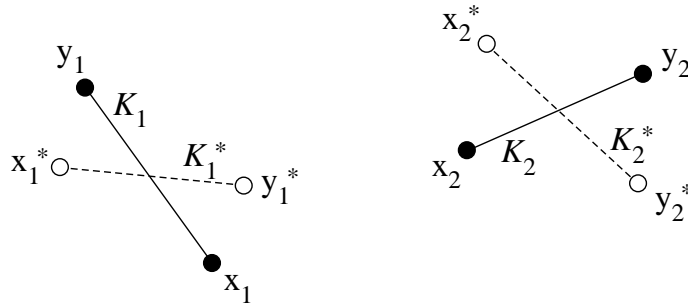

Figure 2. At crossing 1 of two rapidity lines we have spin sites x_1 and y_1 together with dual spin sites x_1^* and y_1^*. At crossing 2 we have positions x_2 and y_2 together with x_1^* and y_1^* oriented similarly.

$$\sinh(2K_1)\sinh(2K_2)\{\langle \sigma_{x_1}\sigma_{x_2}\rangle\langle \sigma_{y_1}\sigma_{y_2}\rangle - \langle \sigma_{x_1}\sigma_{y_2}\rangle\langle \sigma_{y_1}\sigma_{x_2}\rangle\}$$

$$+ \{\langle \sigma_{x_1^*}\sigma_{x_2^*}\rangle^*\langle \sigma_{y_1^*}\sigma_{y_2^*}\rangle^* - \langle \sigma_{x_1^*}\sigma_{y_2^*}\rangle^*\langle \sigma_{y_1^*}\sigma_{x_2^*}\rangle^*\} = 0, \qquad (7)$$

with two arbitrary nearest-neighbour pairs of spins at the sites $\{x_1, y_1\} \neq \{x_2, y_2\}$, and corresponding nearest-neighbour pairs of dual spins at $\{x_1^*, y_1^*\}$ and $\{x_2^*, y_2^*\}$, whereas $\sinh(2K_i)\sinh(2K_i^*) \equiv 1$, $(i = 1, 2)$. Orientations have to be consistent as in the picture, otherwise the plus changes to a minus.

Restricted to Z-invariant Ising models, the quadratic identity reduces to

$$k^2\mathrm{sc}(u_2 - u_1, k')\mathrm{sc}(u_4 - u_3, k')$$
$$\times \{g(u_1, u_2, u_3, u_4, \cdots)\, g(\cdots) - g(u_1, u_2, \cdots)\, g(u_3, u_4, \cdots)\}$$

$$+ \{g^*(u_1, u_3, \cdots)\, g^*(u_2, u_4, \cdots) - g^*(u_1, u_4, \cdots)\, g^*(u_2, u_3, \cdots)\} = 0, \quad (8)$$

with "$\cdots$" short-hand for all other rapidity variables $u_5, u_6, \cdots$, common to all g's and g^*'s (passing between all eight sites).

Knowing $g(u, u, \cdots, u)$ and $g(v, u, \cdots, u)$, with all or all-but-one of the rapidities equal, all other g's can be calculated by recurrence. Therefore, the knowledge of the diagonal and next-to-diagonal pair correlations in the uniform asymmetric ($K \neq \bar{K}$) square-lattice Ising model suffices [6, 7].

Jin has found the scaling form for the general Z-invariant case in the critical regime $k \approx 1$ [5, 7]. From this we can find the first two terms of the susceptibility in several lattices confirming Guttmann's extended lattice-lattice scaling [12, 13].

1.4. Summary of findings

We can make the couplings J and/or the lattice aperiodic. We find the following [5, 6, 14, 15]:

- Periodic lattice with periodic couplings: Periodic $\chi(\mathbf{q})$, with peaks at sites commensurate with reciprocal lattice, becoming sharper and sharper as $T \to T_c$. This includes fully-frustrated cases.
- Periodic lattice with ferromagnetic couplings varying quasiperiodically: Periodic $\chi(\mathbf{q})$, with peaks at reciprocal lattice sites, sharper and sharper as $T \to T_c$
- Periodic lattice with mixed FM and AFM couplings quasiperiodically arranged: Periodic $\chi(\mathbf{q})$, with more and more incommensurate peaks within unit cell as $T \to T_c$
- Quasiperiodic lattice: Quasiperiodic $\chi(\mathbf{q})$, with more and more peaks visible closer to T_c

For Z-invariant lattices, we can evaluate $\chi(\mathbf{q})$ numerically to high accuracy from the recurrence relations for the pair correlations. However, the structure is clearer in density plots [5, 6].

1.5. Generalized Fibonacci Ising lattices

We can assign the couplings according to quasiperiodic sequences in horizontal, vertical, and/or diagonal directions. We can use [14] de Bruijn's generalized Fibonacci

sequences, assigning different couplings according to the sequence of zeros and ones

$$p_j(n) \equiv \lfloor \gamma + (n+1)/\alpha_j \rfloor - \lfloor \gamma + n/\alpha_j \rfloor \tag{9}$$

with

$$\alpha_j \equiv \frac{1}{2}[(j+1) + \sqrt{(j+1)^2 + 4}] \tag{10}$$

We found that such ferromagnetic cases differ very little from periodic cases.

For the mixed ferro/antiferro case, the $\chi(\mathbf{q})$ depends strongly on the sequence chosen, even for the simplest examples just adding signs to the couplings of the Onsager square lattice model by gauge transform [14].

1.6. Pentagrid Ising lattices

Following Korepin, we can use de Bruijn's pentagrid for the rapidity lines leading to a spin model and its dual taking alternating sites of a Penrose tiling [15]. By the quadratic recurrence relations we can compute a big collection of pair correlations. Then, in order to calculate $\chi(\mathbf{q})$, we have developed a novel way of determining the pair probability of local environments on a Penrose tiling, which can also be used once more detailed results for pair correlations in e.g. the eight-vertex model or the chiral Potts model become available. Full details will be published elsewhere [15].

2. Fully-frustrated square-lattice Ising model

There are two common ways to define a fully-frustrated Ising model on a square lattice. One way is the checkerboard version of coloring the faces of the lattice alternatingly black and white and taking three of the couplings around each black square $+J$ and the fourth one $-J$. The other way is to take all horizontal couplings $+J$ and the couplings in vertical columns alternatingly $+J$ and $-J$. This is illustrated in Fig. 3.
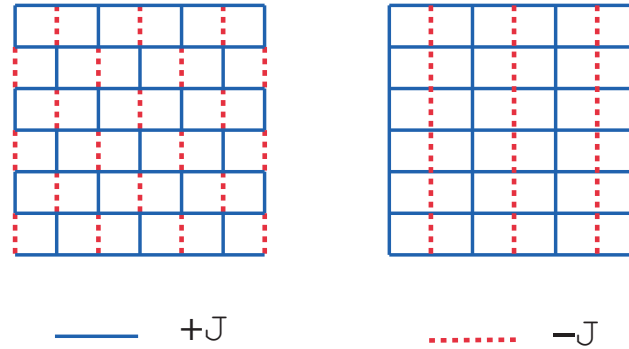

Figure 3.   Two versions of the fully-frustrated square-lattice Ising model with ferromagnetic couplings $+J$ and antiferromagnetic couplings $-J$: (a) Checkerboard version on the left; (b) version periodic in vertical direction on the right.

It must be noted that these two versions do not define fundamentally different models. They are, in fact, related with each other by a simple gauge transformation of flipping the signs of each second horizontal pair of rows of spins, as is indicated in Fig. 4. Of course, there are many other ways of flipping subsets of the spins and one
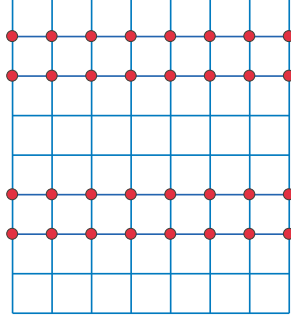
Figure 4. Gauge equivalence of the checkerboard and vertically periodic cases: Flip the signs of the red spins to go from the one to the other, or back.

can define infinitely many other related less regular frustrated models.

There are several approaches to the explicit evaluation of pair correlations and susceptibilities in this model.

2.1. Sum out every other spin

One approach is to first sum out every other spin in order to arrive at an effective Baxter eight-vertex model [16, 17], with diagonal couplings $\hat{J}$ and $\hat{J}'$ and four-spin coupling $\hat{J}_4$. If one does this for the checkerboard case, see also Fig. 5, one finds
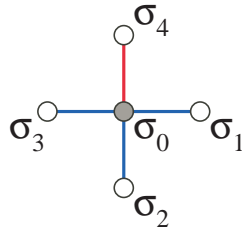
Figure 5. Spin σ_0 to be "decimated out" with its four neighbours.

$$\mathrm{e}^{4\hat{K}} = \frac{1}{\sqrt{2S^2 + 1}}, \quad \mathrm{e}^{4\hat{K}'} = \sqrt{2S^2 + 1}, \quad \mathrm{e}^{4\hat{K}_4} = \frac{S^2 + 1}{\sqrt{2S^2 + 1}} \tag{11}$$

with $\hat{K} \equiv \hat{J}/k_{\mathrm{B}}\hat{T}$, $\hat{K}' \equiv \hat{J}'/k_{\mathrm{B}}\hat{T}$ and $\hat{K}_4 \equiv \hat{J}_4/k_{\mathrm{B}}\hat{T}$, while using the short-hand notation

$$S \equiv \sinh \frac{2J}{k_{\mathrm{B}}T} \equiv \sinh 2K \tag{12}$$

In terms of Baxter's a, b, c, d, this means

$$a = b = \sqrt{S^2 + 1}, \quad c = 1, \quad d = \sqrt{2S^2 + 1} \tag{13}$$

Note that only diagonal and four-spin interactions are left. All other interactions cancel out.

This determines the mapping of Boltzmann weights. For correlation functions we have also to find the mapping of the spins involved. Spins σ_1, σ_2, σ_3, σ_4 in Fig. 5 are left unchanged. However, the central spin σ_0 in Fig. 5 has to be replaced after "decimation summation" according to

$$\sigma_0 \quad \longrightarrow \quad \frac{S\left(\sigma_1 + \sigma_3 + \sigma_2 - \sigma_4\right)}{2\sqrt{S^2 + 1}} \left(1 - \frac{S^2(1 - \sigma_1\sigma_3\sigma_2\sigma_4)}{2(2S^2 + 1)}\right) \tag{14}$$

2.2. Partial duality transform

After this, we first flip the sign of each $\sigma_2\sigma_4$, interchanging $a \leftrightarrow d$, $b \leftrightarrow c$. We then have

$$e^{4\tilde{K}} = e^{4\tilde{K}'} = \sqrt{2S^2 + 1}, \qquad e^{4\tilde{K}_4} = \frac{\sqrt{2S^2 + 1}}{S^2 + 1}, \qquad S \equiv \sinh\frac{2J}{k_{\mathrm{B}}T} \tag{15}$$

Next, we apply a partial Kramers–Wannier duality transform on every other spin [16, 17]. This leads to an Ashkin–Teller model, with new spins τ on the same positions as a quarter of the original spins σ and three quarters of the original positions empty. All original spins σ are now expressed in terms of σ's and τ^*'s (disorder variables, equal to other original σ's).

Finally, replacing $\sigma \longrightarrow \sigma\tau$, $\tau \longrightarrow \tau$, $\sigma^* \longrightarrow \sigma^*$, $\tau^* \longrightarrow \sigma^*\tau^*$, the Ashkin–Teller model factorizes into two Ising models, at dual temperatures $K_\sigma = \beta_\sigma J_0$, $K_\tau = \beta_\sigma^* J_0$, with

$$\sinh(2K_\sigma) = \frac{S^2}{S^2 + 1 + \sqrt{2S^2 + 1}} = \frac{1}{\sinh(2K_\tau)} \equiv \sqrt{k} \tag{16}$$

2.3. Difference equations for the correlation functions

Writing $C(m, n) \equiv \langle \sigma_{0,0}\sigma_{m,n} \rangle_{K_\sigma}$ and $\bar{C}(m, n) \equiv \langle \tau_{0,0}\tau_{m,n} \rangle_{K_\tau}$ for the two resulting independent Ising models, we can determine these recursively [4, 7] from

$$[C(m, n{+}1)C(m, n{-}1) - C(m, n)^2]$$
$$+ k\left[\bar{C}(m{+}1, n)\bar{C}(m{-}1, n) - \bar{C}(m, n)^2\right] = 0 \tag{17}$$

$$[C(m{+}1, n)C(m{-}1, n) - C(m, n)^2]$$
$$+ k\left[\bar{C}(m, n{+}1)\bar{C}(m, n{-}1) - \bar{C}(m, n)^2\right] = 0 \tag{18}$$

$$[C(m, n)C(m{+}1, n{+}1) - C(m{+}1, n)C(m, n{+}1)] =$$
$$k\left[\bar{C}(m, n)\bar{C}(m{+}1, n{+}1) - \bar{C}(m{+}1, n)\bar{C}(m, n{+}1)\right] \tag{19}$$

$$\sqrt{k}\left[C(m{+}1, n)\bar{C}(m{-}1, n) + C(m{-}1, n)\bar{C}(m{+}1, n)\right.$$
$$+ C(m, n{+}1)\bar{C}(m, n{-}1) + C(m, n{-}1)\bar{C}(m, n{+}1)]$$
$$= (k{+}1)\, C(m, n)\bar{C}(m, n) \tag{20}$$

with

$$C(0,0) = \bar{C}(0,0) = 1, \quad C(1,0) + \sqrt{k}\,\bar{C}(0,1) = \sqrt{k+1} \tag{21}$$
$$C(m,n) = C(n,m) = C(|m|,|n|)$$
$$\bar{C}(m,n) = \bar{C}(n,m) = \bar{C}(|m|,|n|) \tag{22}$$

2.4. Pair correlation function of fully-frustrated model

Finally, the pair correlation functions of the original model of Fig. 3 (a) are given by

$$\langle \sigma_{kl}\,\sigma_{k+2m,l+2n} \rangle = (-1)^n\,C(m,n)\,\bar{C}(m,n) \tag{23}$$
$$\langle \sigma_{kl}\,\sigma_{k+2m-1,l+2n-1} \rangle = 0 \tag{24}$$
$$\langle \sigma_{kl}\,\sigma_{k+2m-1,l+2n} \rangle$$
$$= \frac{(-1)^n\,S}{2\sqrt{2S^2+1}}\,(C(m-1,n)\,\bar{C}(m,n) + C(m,n)\,\bar{C}(m-1,n)) \tag{25}$$
$$\langle \sigma_{kl}\,\sigma_{k+2m,l+2n-1} \rangle$$
$$= \frac{\pm(-1)^n\,S}{2\sqrt{2S^2+1}}\,(C(m,n-1)\,\bar{C}(m,n) + C(m,n)\,\bar{C}(m,n-1)) \tag{26}$$

In the last equation we have to choose plus $(+)$ if $\min(k+l, k+2m+l+2n-1) =$ even and minus $(-)$ if $\min(k+l, k+2m+l+2n-1) =$ odd. Therefore, its contribution to $\chi(q_x, q_y)$ averages out to zero. If we had started from the vertically periodic case Fig. 3 (b), we must omit all three $(-1)^n$ factors in the above.

Obviously both models have a periodic $\chi(q_x, q_y)$ with commensurate peaks only, as can be illustrated with density plots. Using the results of this subsection it is also not difficult to derive a long series expansion along the lines of the Ising model work [2].

2.5. Other approaches

The square-lattice fully-frustrated Ising model is dual to the Ising model in field $i\pi k_B T/2$ introduced by Lee and Yang. For this case it in convenient to view a square of four spins as a vertex of a 16-vertex model and then to apply gauge transforms [11]. Now the model decouples as a product of two Ising models at the same temperature. For the special case of the square-lattice dimer model these two models are at the critical temperature and the explicit formula for the two-point function has been published [18] and the monomer-monomer correlation function and its Fourier transform $\chi(q_x, q_y)$ have been studied in some detail by Kong [19].

2.6. Final remarks: Checkerboard Ising and chiral Potts

The physical free-fermion model corresponds to a checkerboard Ising model [10] with real or imaginary elliptic modulus k. In general, the correlation functions satisfy a 2-by-2 matrix generalization [20] of the difference equations used above, allowing incommensurate solutions.

For the two-dimensional integrable N-state chiral Potts model a similar mapping to an N-state generalization of the free-fermion 8-vertex model exists [21]. This model is a most natural generalization of Onsager's two-dimensional Ising model to more than two states per spin. The pair-correlation functions have not yet been solved for this model; only the conjecture for its order parameters (one-point functions) has been proved recently by Baxter [22]. The model is on a submanifold in the commensurate phase of the more general chiral Potts model. The full chiral Potts model also has incommensurate phases. However, Jin's results [23] do not seem to support the presence of a Lifshitz point in the classical two-dimensional model.

3. Overlapping unit cells

Gummelt [24] motivated by physical considerations, has proposed a description of quasicrystals in terms of overlapping unit cells, with the regular Penrose tiling described by overlappings of decorated decagons. We have used a multigrid method based on de Bruijn's work to produce a new example of 3-dimensional overlapping unit cells, quasiperiodic in two directions and periodic in the third. Full details are presented elsewhere [25].

As our construction is based on a multigrid of five grids of parallel planes, resulting from a projection of the five-dimensional hypercubic lattice into three dimensions, one may construct aperiodic integrable three-dimensional models with spectral variables living on the grid planes and with Boltzmann weights satisfying the tetrahedron equations, generalizing the pentagrid Ising model construction.

References

[1] Wu T T, McCoy B M, Tracy C A and Barouch E 1976 Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region *Phys. Rev.* B **13** 316–374
[2] Orrick W P, Nickel B G, Guttmann A J and Perk J H H 2001 Critical behavior of the two-dimensional Ising susceptibility *Phys. Rev. Lett.* **86** 4120–3
[3] Orrick W P, Nickel B, Guttmann A J and Perk J H H 2001 The susceptibility of the square lattice Ising model: New developments *J. Stat. Phys.* **102** 795–841
[4] Perk J H H 1980 Quadratic identities for Ising model correlations *Phys. Lett.* A **79** 3–5
[5] Au-Yang H, Jin B-Q and Perk J H H 2001 Wavevector-dependent susceptibility in quasiperiodic Ising models *J. Stat. Phys.* **102** 501–43
[6] Au-Yang H and Perk J H H 2002 Wavevector-dependent susceptibility in aperiodic planar Ising models *MathPhys Odyssey 2001: Integrable Models and Beyond* ed Kashiwara M and Miwa T (Birkhäuser, Boston) pp 1–21

[7] Au-Yang H and Perk J H H 2002 Correlation functions and susceptibility in the Z-invariant Ising model *MathPhys Odyssey 2001: Integrable Models and Beyond* ed Kashiwara M and Miwa T (Birkhäuser, Boston) pp 23–48

[8] Caselle M, Hasenbusch M, Pelissetto A and Vicari E 2002 Irrelevant operators in the two-dimensional Ising model *J. Phys. A: Math. Gen.* **35** 4861–88

[9] Baxter R J 1978 Solvable eight vertex model on an arbitrary planar lattice *Phil. Trans. R. Soc. Lond.* A **289** 315–46

[10] Baxter R J 1986 Free-fermion, checkerboard and Z-invariant lattice models in statistical mechanics *Proc. R. Soc. Lond.* A **404** 1–33

[11] Au-Yang J and Perk J H H 1987 Critical correlations in a Z-invariant inhomogeneous Ising model *Physica* A **144** 44–104

[12] Au-Yang H and Perk J H H 2002 New results for susceptibilities in planar Ising models *Int. J. Mod. Phys.* B **16** 2089–95

[13] Au-Yang H and Perk J H H 2003 Susceptibility calculations in periodic and quasiperiodic planar Ising models *Physica* A **321** 81–9

[14] Au-Yang H and Perk J H H 2005 Q-dependent susceptibilities in ferromagnetic quasiperiodic Z-invariant Ising models *Preprint*

[15] Au-Yang H and Perk J H H 2005 Q-dependent susceptibilities in Z-invariant pentagrid Ising models *Preprint* cond-mat/0409557

[16] Forgács G 1980 Ground-state correlations and universality in two-dimensional fully frustrated Ising model *Phys. Rev.* B **22** 4473–80

[17] Forgács G and Fradkin E 1981 Anisotropy and marginality in the two-dimensional fully frustrated Ising model *Phys. Rev.* B **23** 3442-7

[18] Au-Yang H and Perk J H H 1984 Ising correlations at the critical temperature *Phys. Lett.* A **104A** 131–4

[19] Kong X-P 1987 Wave-Vector Dependent Susceptibility of the Two-Dimensional Ising Model *Ph D Thesis* (State University of New York at Stony Brook)

[20] Perk J H H 1980 Equations of motion for the transverse correlations of the one-dimensional XY-model at finite temperature *Phys. Lett.* A **79** 1–2

[21] Baxter R J, Perk J H H and Au-Yang H 1988 New solutions of the star-triangle relations for the chiral Potts model *Phys. Lett.* A **128** 138–42

[22] Baxter R J 2005 The order parameter of the chiral Potts model *J. Stat. Phys.* **120** 1–36

[23] Jin B-Q 2001 Some Aspects of the Chiral Potts Model and the Ising Model *Ph D Thesis* (Oklahoma State University, Stillwater)

[24] Gummelt P 1996 Penrose tilings as coverings of congruent decagons *Geometriae Dedicata* **62** 1–17

[25] Au-Yang H and Perk J H H 2005 Overlapping unit cells in 3-d quasicrystal structure *Preprint* cond-mat/0507117